

Pivotal Quantities with Arbitrary Small Skewness

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Abstract

In this paper we present randomization methods to enhance the accuracy of the central limit theorem (CLT) based inferences about the population mean μ . We introduce a broad class of randomized versions of the Student t -statistic, the classical pivot for μ , that continue to possess the pivotal property for μ and their skewness can be made arbitrarily small, for each fixed sample size n . Consequently, these randomized pivots admit CLTs with smaller errors. The randomization framework in this paper also provides an explicit relation between the precision of the CLTs for the randomized pivots and the volume of their associated confidence regions for the mean for both univariate and multivariate data. This property allows regulating the trade-off between the accuracy and the volume of the randomized confidence regions discussed in this paper.

1 Introduction

The CLT is an essential tool for inferring on parameters of interest in a nonparametric framework. The strength of the CLT stems from the fact that, as the sample size increases, the usually unknown sampling distribution of a pivot, a function of the data and an associated parameter, approaches the standard normal distribution. This, in turn, validates approximating the percentiles of the sampling distribution of the pivot by those of the normal distribution, in both univariate and multivariate cases.

The CLT is an approximation method whose validity relies on large enough samples. In other words, the larger the sample size is, the more accurate the inference, about the parameter of interest, based on the CLT will be. The accuracy of the CLT can be evaluated in a number ways. Measuring the distance between the sampling

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distribution of the pivot and the standard normal distribution is the common feature of these methods. Naturally, the latter distance is a measure of the error of the CLT. The most well known methods of evaluating the CLT's error are Berry-Esséen inequalities and Edgeworth expansions. These methods have been extensively studied in the literature and many contributions have been made to the area (cf., for example, Barndorff-Nielsen and Cox [3], Bentkus *et al.* [5], Bentkus and Götze [6], Bhattacharya and Rao [7], DasGupta [9], Hall [12], Petrov [16], Senatov [17], Shao [20] and Shorack [21]).

Despite their differences, the Berry-Esséen inequality and the Edgeworth expansion, when the data have a finite third moment, agree on concluding that, usually, the CLT is in error by a term of order $O(1/\sqrt{n})$, as $n \rightarrow +\infty$, where n is the sample size. In the literature, the latter asymptotic conclusion is referred to as the first order accuracy or efficiency of the CLT.

Achieving more accurate CLT based inferences requires alternative methods of extracting more information, about the parameter of interest, from a given sample that may not be particularly large.

In this paper we introduce a method to significantly enhance the accuracy of confidence regions for the population mean via creating new pivots for it based on a given set of data. More precisely, by employing appropriately chosen random weights, we construct new randomized pivots for the mean. These randomized pivots are more symmetrical than their classical counterpart the Student t -statistic and, consequently, they admit CLTs with smaller errors for both univariate and multivariate data. In fact, by choosing the random weights appropriately, we will see that the CLTs for the introduced randomized pivots, under some conventional conditions, can already be second order accurate (see Sections 3 and 6).

The randomization framework in this paper can be viewed not only as an alternative to the inferences based on the classical CLT, but also to the bootstrap. The bootstrap, introduced by Efron [10], is a method that also tends to increase the accuracy of CLT based inferences (cf., e.g., Hall [12] and Singh [22]). The bootstrap relies on repeatedly re-sampling from a given data set (see, for example, Efron and Tibshirani [11]).

The methodology introduced in this paper, on the other hand, reduces the error of the CLT in a customary fashion, in both univariate and multivariate cases, and it does not require re-sampling from the given data (see Remark 5.2 below for a brief comparison between the randomization approach of this paper and the bootstrap).

For confidence regions based on CLTs to capture a parameter of interest, in addition to the accuracy, it is desirable to also address their volume.

In this paper we also address the volume of the resulting confidence regions based on our randomized pivotal quantities in both univariate and multivariate cases. In the randomization framework of this paper, and in view of the CLTs for the randomized pivots introduced in it, studying the volume of the resulting randomized confidence regions for the mean is rather straightforward. This, in

turn, enables one to easily trace the effect of the reduction in the error, i.e., the higher accuracy, on the volume of the resulting confidence regions. As a result, one will be able to regulate the trade-off between the precision and the volume of the randomized confidence regions (see Section 4, Section 6 and Appendix I).

The rest of this paper is organized as follows. In Section 2 we introduce the new randomized pivots for the mean of univariate data. In Section 3 we use Edgeworth expansions to explain how the randomization techniques introduced in Section 2 result in a higher accuracy of the CLT. In Section 4, for univariate data, we investigate the length of the confidence intervals that result from the use of the randomized pivots introduced in Section 2. Extensions of the randomization techniques of Section 2 to classes of triangular random weights are presented in Section 5. Generalization of the results in Sections 2 and 3 to vector valued data are presented in Section 6.

2 Randomized pivots with higher accuracy

Let X, X_1, \dots, X_n , $n \geq 1$, be i.i.d. random variables with $E_X|X_1|^3 < +\infty$, $\mu := E_X(X_1)$ and $\sigma_X^2 := Var_X(X_1) > 0$. The Student t -statistic, the classical pivot for μ , is defined as:

$$t_n := \sum_{i=1}^n (X_i - \mu) / (S_n \sqrt{n}), \quad (2.1)$$

where $S_n^2 = \sum_{i=1}^n (X_i - \bar{X}_n)^2 / n$ and \bar{X}_n are the sample variance and the sample mean, respectively. Under the assumption $E_X|X_1|^3 < +\infty$, the Berry-Esséen inequality and the Edgeworth expansion unanimously assert that, without restricting the class of distributions of the data, t_n converges in distribution to standard normal at the rate $O(1/\sqrt{n})$, i.e., the CLT for t_n is first order accurate. We are now to improve upon the accuracy of t_n by using a broad class of random weights. The improvement will result from replacing the pivot t_n by randomized versions of it that continue to serve as pivots for μ .

We now define the aforementioned randomized pivots for μ , as follows:

$$g_n^w(\theta) := \sum_{i=1}^n (w_i - \theta)(X_i - \mu) / (S_n \sqrt{n E_w(w_1 - \theta)^2}), \quad (2.2)$$

where w 's are some random weights and θ , to which we refer as the window, is a real valued constant. The weights w 's and the window constant θ are to be chosen according to either one of the following two scenarios, namely, Method I and Method II.

Method I: Non-centered weights

To construct the randomized pivot $g_n^w(\theta)$ in this scenario, we let the weights w_1, \dots, w_n be a *random sample* with $E_w|w_1|^3 < +\infty$. Moreover, these weights should be *independent* from the data X_1, \dots, X_n . The window constant θ , should be chosen in such a way that it satisfies the following two properties:

- (i) $\theta \neq E_w(w_1)$,
- (ii) $\text{SRF}^w(\theta) := E_w(w_1 - \theta)^3 / (E_w(w_1 - \theta)^2)^{3/2} = \delta$,

where δ is a given number such that $|\delta|$ can be arbitrary small or zero.

The notation SRF is an abbreviation for **S**kewness **R**educing **F**actor (see (3.1) below for a justification for this notation).

Remark 2.1. *The weights w 's in Method I can be generated, independently from the data, using some statistical software. This remark is applicable to all randomized pivots discussed in this paper.*

Discussion of Method I: When the weights have a skewed distribution

In terms of the error of the CLT, an ideal realization of condition (ii) of Method I could be when the weights w 's have a skewed distribution and the window constant θ is a real root for the cubic equation $E_w(w_1 - \theta)^3 = 0$, i.e., when $\delta = 0$. Condition (ii) of Method I is so that it also allows generating the w 's from skewed distributions and finding a window constant θ such that $\theta \neq E_w(w_1)$ and $\text{SRF}^w(\theta)$ is close enough to zero. Hence, when $\delta \neq 0$, but $|\delta|$ is chosen to be small, then θ does not necessarily have to be a root of the equation $E_w(w_1 - \theta)^3 = 0$.

As it can be inferred from the results in Section 3, the closer the value of the $\text{SRF}^w(\theta)$ is to zero, the smaller the error of the CLT for $g_n^w(\theta)$, as in (2.2), will be.

Discussion of Method I: When the weights are symmetrical about their mean

When the w 's are generated from a distribution that is symmetrical about its mean, in view of Method I, a refinement can be achieved by taking the window constant θ to be close to $E_w(w_1)$ but not equal to it. This choice of θ will result in $\text{SRF}^w(\theta)$ that are not exactly zero, but can be arbitrarily close to it.

Method II that follows can also be used to construct a more accurate randomized pivot $g_n^w(\theta)$, as defined in (2.2), via generating the random weights from some symmetrical distributions.

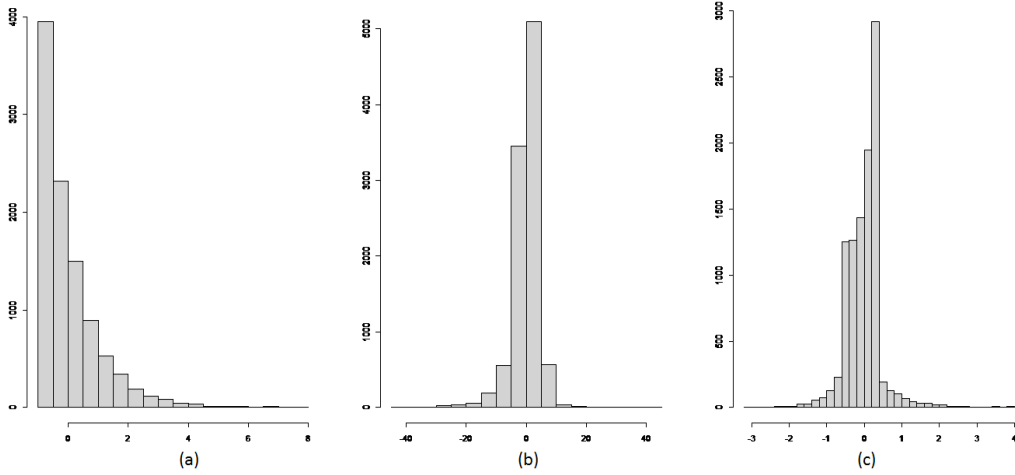


Figure 1: (Illustration of the effect of Method I on univariate data)

Panel (a) is the frequency histogram of the original centered data $(X_i - 1)$, $1 \leq i \leq 10000$, where X_i 's are i.i.d. Exponential(1) with empirical Pearson's measure of skewness equal to 1.98. Panel (b) is the frequency histogram of the randomized data $(w_i - 9.3)(X_i - 1)$, where the weights w_i 's are i.i.d. $\chi^2(7)$, $\theta = 9.3$ and $\text{SRF}^w(9.3) \approx -0.622$, with empirical Pearson's measure of skewness equal to -1.29 . Panel (c) is the frequency histogram of the randomized data $(w_i - 0.58)(X_i - 1)$, where the weights w_i 's are i.i.d. Bernoulli(1/3), $\theta = 0.58$ and $\text{SRF}^w(0.58) \approx -0.7$, with empirical Pearson's measure of skewness equal to -1.34 .

Method II: Symmetrical and centered weights

In this scenario, we let the weights w_1, \dots, w_n be a *random sample* with a symmetrical (about its mean) distribution and $E_w|w_1|^3 < +\infty$. Moreover, we assume that the weights are *independent* from the data X_1, \dots, X_n and we take the window constant θ to be equal to the mean of the random weights, i.e., $\theta = E_w(w_1)$.

Taking $\theta = E_w(w_1)$ together with the symmetry of the distribution of the weights, imply that, in the scenario of Method II, we have $\text{SRF}^w(\theta) = 0$, where $\text{SRF}^w(\theta)$ is as defined in (ii) of Method I.

Comparing Method I to Method II

Using the randomized pivot $g_n^w(\theta)$, as in (2.2), and generating its associated random weights w 's according to either Method I or Method II, can result in a significant refinement in inferring about μ . The reason for this claim is given in Section 3.

In spite of the higher accuracy of $g_n^w(\theta)$, provided by both Method I and Method II, we emphasize that the former is more desirable. This is so since, in both univariate and multivariate cases, Method I yields randomized confidence regions for μ whose volumes shrink to zero as the sample size n increases to infinity (see (4.3) and Appendix I). Method II, on the other hand, fails to yield shrinking confidence

regions. In fact, choosing the weights w 's for the pivot $g_n^w(\theta)$ under the scenario of Method II, yields confidence regions for μ whose volumes, as $n \rightarrow +\infty$, approach a limiting distribution rather than vanishing (see (4.4) and Table 3 below).

Remark 2.2. *The term $nE_w(w_1 - \theta)^2$ in the denominator of $g_n^w(\theta)$, as in (2.2), under both Methods I and II, can, equivalently, be replaced by $\sum_{j=1}^n (w_j - \theta)^2$.*

In the above description of different weights in Methods I and II, we excluded the case when the weights w 's have a skewed distribution and $\theta = E_w(w_1)$. This case was omitted since, in general, it does not necessarily provide a refinement in the CLT for the resulting randomized pivot $g_n^w(\theta)$, nor does it result in confidence regions whose volumes shrink to zero, as the sample size increases.

3 Error of the CLT for $g_n^w(\theta)$ under Methods I and II

The main idea behind Methods I and II is to transform the classical pivot t_n , as in (2.1), to $g_n^w(\theta)$, as in (2.2), that has a smaller *skewness*. To further develop the idea, we first note that $g_n^w(\theta)$ is governed by the joint distribution of the data X and the weights w 's. In view of this observation, we let $P_{X,w}$ stand for the joint distribution of the data and the weights, and we represent its associated mean by $E_{X,w}$. Recalling now that in both Method I and Method II the weights are independent from the data, we conclude that $P_{X,w} = P_X \cdot P_w$ and, consequently, $E_{X,w} = E_X \cdot E_w$.

Now observe that

$$E_{X,w}((w_1 - \theta)(X_1 - \mu)) = E_X(X_1 - \mu)E_w(w_1 - \theta) = 0 \cdot E_w(w_1 - \theta) = 0.$$

In view of the preceding observation, we now obtain the skewness of the random variables $(X - \mu)(w - \theta)$, under both Methods I and II, as follows:

$$\begin{aligned} \text{skewness of } (X - \mu)(w - \theta) &= \frac{E_{X,w}((X_1 - \mu)(w_1 - \theta))^3}{\{E_{X,w}((X_1 - \mu)(w_1 - \theta))^2\}^{3/2}} \\ &= \left(\frac{E_w(w_1 - \theta)^3}{\{E_w(w_1 - \theta)^2\}^{3/2}} \right) \left(\frac{E_X(X_1 - \mu)^3}{\sigma_X^3} \right) \\ &= \text{SRF}^w(\theta) \left(\frac{E_X(X_1 - \mu)^3}{\sigma_X^3} \right). \end{aligned} \quad (3.1)$$

The second term of the product on the r.h.s. of (3.1), i.e., $E_X(X_1 - \mu)^3 / \sigma_X^3$, is the skewness of the original data. The closer it is to zero the nearer the sampling

distribution of t_n , as defined in (2.1), will be to the standard normal. However, one usually has no control over the skewness of the original data. The idea in Methods I and II is to incorporate the random weights w 's and to appropriately choose a window constant θ in such a way that $|\text{SRF}^w(\theta)|$ is arbitrarily small. This, in view of (3.1), will result in smaller skewness of the random variables $(X - \mu)(w - \theta)$ (see also Appendix II for the effect of the skewness reduction methods on vector valued data). The latter property, in turn, under appropriate conditions, can result in the second order accuracy of the CLTs for $g_n^w(\theta)$, as defined in (2.2), under both Methods I and II. The accuracy of $g_n^w(\theta)$ is to be discussed later in this section in the univariate case and, in Section 6 in the multivariate case.

In view of (3.1), it is now easy to appreciate that when θ is chosen in such a way that $\text{SRF}^w(\theta) = 0$, then the skewness of $(X - \mu)(w - \theta)$ will be exactly zero. The latter case can happen under Method I when the distribution of the w 's is skewed and the cubic equation $E_w(w - \theta)^3 = 0$ has at least one real root and θ is taken to be one of these real roots. The other way to make $\text{SRF}^w(\theta)$ equal to zero is when the weights w 's have a symmetrical distribution and $\theta = E_w(w)$, i.e., Method II. However, when Method II is used to construct $g_n^w(\theta)$, having an $\text{SRF}^w(\theta)$ that is exactly zero, as it was already mentioned in Section 2, will come at the expense of having confidence regions for μ whose volumes do not vanish (see Section 4 and Appendix I).

Edgeworth expansions for $g_n^w(\theta)$ in view of Methods I and II

We use Edgeworth expansions to illustrate the higher accuracy of the CLT for the randomized pivot $g_n^w(\theta)$, as in (2.2), under Methods I and II, as compared to that of the classical CLT for the pivot t_n , as in (2.1). Edgeworth expansions are used in our reasoning below since they provide a direct link between the skewness of a pivotal quantity and the error admitted by its CLT.

In order to state the Edgeworth expansion for the sampling distribution $P_{X,w}(g_n^w(\theta) \leq t)$, for all $t \in \mathbb{R}$, we first define

$$Z_n^w(\theta) := \sum_{i=1}^n (w_i - \theta)(X_i - \mu) / \sqrt{n\sigma_X^2 E_w(w_1 - \theta)^2}. \quad (3.2)$$

Also, we consider arbitrary positive ϵ and ϵ_1 , and we let $\epsilon_2 > 0$ be so that $\Phi(t + \epsilon) - \Phi(t) \leq \epsilon_2$, where Φ stands for the standard normal distribution function.

In view of the above setup, we now write the following approximation.

$$\begin{aligned}
& -\left(\frac{\epsilon_1}{\epsilon}\right)^2 - P_X(|S_n^2 - \sigma_X^2| > \epsilon_1^2) + P_{X,w}(Z_n^w(\theta) \leq t - \epsilon) - \Phi(t - \epsilon) - \epsilon_2 \\
& \leq P_{X,w}(g_n^w(\theta) \leq t) - \Phi(t) \\
& \leq \left(\frac{\epsilon_1}{\epsilon}\right)^2 + P_X(|S_n^2 - \sigma_X^2| > \epsilon_1^2) + P_{X,w}(Z_n^w(\theta) \leq t + \epsilon) - \Phi(t + \epsilon) + \epsilon_2.
\end{aligned} \tag{3.3}$$

Under the assumption $E_X|X_1|^3 < +\infty$, from Baum and Katz [4], we conclude that, as $n \rightarrow +\infty$,

$$P(|S_n^2 - \sigma_X^2| > \epsilon_1^2) = O(1/(\sqrt{n} \log^2 n)).$$

By virtue of this result, we conclude that replacing $g_n^w(\theta)$ by $Z_n^w(\theta)$ produces an error that approaches zero at the rate $o(1/\sqrt{n})$, as $n \rightarrow +\infty$.

Combining now the preceding conclusion with (3.3) and letting $\varepsilon := (\epsilon_1/\epsilon)^2 + \epsilon_2$, we arrive at

$$\begin{aligned}
& -\varepsilon - o(1/\sqrt{n}) + P_{X,w}(Z_n^w(\theta) \leq t - \epsilon) - \Phi(t - \epsilon) \\
& \leq P_{X,w}(g_n^w(\theta) \leq t) - \Phi(t) \\
& \leq \varepsilon + o(1/\sqrt{n}) + P_{X,w}(Z_n^w(\theta) \leq t + \epsilon) - \Phi(t + \epsilon).
\end{aligned} \tag{3.4}$$

The preceding relation implies the asymptotic equivalence of

$$(P_{X,w}(g_n^w(\theta) \leq t) - \Phi(t)) \quad \text{and} \quad (P_{X,w}(Z_n^w(\theta) \leq t) - \Phi(t))$$

up to an error of order $o(1/\sqrt{n})$. In view of this equivalence and also recalling that in both Methods I and II the weights have a finite third moment, we write a one-term Edgeworth expansion for $P_{X,w}(Z_n^w(\theta) \leq t)$, $t \in \mathbb{R}$, as follows:

$$\begin{aligned}
& P_{X,w}(Z_n^w(\theta) \leq t) - \Phi(t) \\
& = -\left(\frac{\phi(t)H_1(t)}{3!\sqrt{n}}\right) \left(\text{SRF}^w(\theta)\right) \left(\frac{E_X(X_1 - \mu)^3}{\sigma_X^3}\right) + o(1/\sqrt{n}),
\end{aligned} \tag{3.5}$$

where ϕ is the density function of the standard normal distribution and $H_1(t) = t^2 - 1$.

Under the condition $E_X|X_1|^3 < +\infty$, the following (3.6) and (3.8) are the respective counterparts of the approximations (3.4) and (3.5) for the classical t_n , as in (2.1), and they read as follows:

$$\begin{aligned}
& -\varepsilon - o(1/\sqrt{n}) + P_X(Z_n \leq t - \epsilon) - \Phi(t - \epsilon) \\
& \leq P_X(t_n \leq t) - \Phi(t) \\
& \leq \varepsilon + o(1/\sqrt{n}) + P_X(Z_n \leq t + \epsilon) - \Phi(t + \epsilon),
\end{aligned} \tag{3.6}$$

where

$$Z_n := \sum_{i=1}^n (X_i - \mu) / \sqrt{n\sigma_X^2}, \quad (3.7)$$

and

$$P_X(Z_n \leq t) - \Phi(t) = -\left(\frac{\phi(t)H_1(t)}{3!\sqrt{n}}\right)\left(\frac{E_X(X_1 - \mu)^3}{\sigma_X^3}\right) + o(1/\sqrt{n}). \quad (3.8)$$

A comparison between the expansions (3.8) and (3.5) shows how incorporating the weights w 's and their associated window θ , as specified in Methods I and II, results in values of $P_{X,w}(g_n^w(\theta) \leq t)$ which are closer to the standard normal distribution $\Phi(t)$ than those of $P_X(t_n \leq t)$. More precisely, under Methods I and II, having an $\text{SRF}^w(\theta)$, such that $|\text{SRF}^w(\theta)|$ is small or negligible, results in smaller or negligible values of the skewness of $g_n^w(\theta)$, as defined in (2.2). The latter reduction of the skewness, when $|\text{SRF}^w(\theta)|$ is negligible, by virtue of (3.4) and (3.5), yields a one-term Edgeworth expansion for the sampling distribution of $g_n^w(\theta)$ whose magnitude of error is $o(1/\sqrt{n})$ rather than $O(1/\sqrt{n})$. On the other hand, in view of (3.6) and (3.8), the rate of convergence of the CLT for the classical t_n , as in (2.1), is of order $O(1/\sqrt{n})$.

In order to further elaborate on the refinement provided by the skewness reduction approach provided by Methods I and II above, we now assume that the data X and the weights w both have a finite fourth moment. In addition to the latter assumption, we also assume that the data X satisfy Cramér's condition that $\limsup_{|t| \rightarrow +\infty} |E_X(\exp\{itX_1\})| < 1$. Cramér's condition is required for the sampling distributions $P_{X,w}(g_n^w(\theta) \leq t)$ and $P_X(t_n \leq t)$ to admit two-term Edgeworth expansions.

It is noteworthy that typical examples of distributions for which Cramér's condition holds true are those with a proper density (cf. Hall [12]).

Once again here, replacing $g_n^w(\theta)$ by $Z_n^w(\theta)$, as in (2.2) and (3.2), generates the error term $P_X(|S_n^2 - \sigma_X^2| > \epsilon_1^2)$, where ϵ_1 is an arbitrary small positive constant. In view of our moment assumption at this stage, $E_X|X_1|^4 < +\infty$, from Baum and Katz [4] we conclude that, as $n \rightarrow +\infty$

$$P_X(|S_n^2 - \sigma_X^2| > \epsilon_1^2) = o(1/n). \quad (3.9)$$

Hence, replacing $g_n^w(\theta)$ by $Z_n^w(\theta)$ generates an error of order $o(1/n)$. By virtue of the latter conclusion, an argument similar to the one used to derive (3.4), yields

$$\begin{aligned} & -\varepsilon - o(1/n) + P_{X,w}(Z_n^w(\theta) \leq t - \epsilon) - \Phi(t - \epsilon) \\ & \leq P_{X,w}(g_n^w(\theta) \leq t) - \Phi(t) \\ & \leq \varepsilon + o(1/n) + P_{X,w}(Z_n^w(\theta) \leq t + \epsilon) - \Phi(t + \epsilon). \end{aligned} \quad (3.10)$$

Also, a similar argument to (3.6) yields

$$\begin{aligned}
& -\varepsilon - o(1/n) + P_X(Z_n \leq t - \varepsilon) - \Phi(t - \varepsilon) \\
& \leq P_X(t_n \leq t) - \Phi(t) \\
& \leq \varepsilon + o(1/n) + P_X(Z_n \leq t + \varepsilon) - \Phi(t + \varepsilon),
\end{aligned} \tag{3.11}$$

where Z_n is as defined in (3.7).

The approximation result (3.10) implies that $g_n^w(\theta)$ and $Z_n^w(\theta)$ are equivalent up to an error of order $o(1/n)$ and (3.11) yields the same conclusion for t_n and Z_n . By virtue of the latter two equivalences, we now write two-term Edgeworth expansions for $P_{X,w}(Z_n^w(\theta) \leq t)$ and $P_X(Z_n \leq t)$, $t \in \mathbb{R}$, as follows:

$$\begin{aligned}
& P_{X,w}(Z_n^w(\theta) \leq t) - \Phi(t) \\
& = -\phi(t) \left\{ \frac{H_1(t)}{3!\sqrt{n}} \left(\text{SRF}^w(\theta) \right) \left(\frac{E_X(X_1 - \mu)^3}{\sigma_X^3} \right) \right. \\
& + \frac{H_2(t)}{4!n} \left\{ \left(\frac{E_w(w_1 - \theta)^4}{E_w^2(w_1 - \theta)^2} \right) \left(\frac{E_X(X_1 - \mu)^4}{\sigma_X^4} \right) - 3 \right\} \\
& + \frac{H_3(t)}{2(3!)^2n} \left(\text{SRF}^w(\theta) \right)^2 \left(\frac{E_X(X_1 - \mu)^3}{\sigma_X^3} \right)^2 \Big\} \\
& + o(1/n),
\end{aligned} \tag{3.12}$$

where $H_1(t)$ is as in (3.5), $H_2(t) = t^3 - 3t$ and $H_3(t) = t^5 - 10t^3 + 15t$.

As to Z_n , it admits the following two-term Edgeworth expansion.

$$\begin{aligned}
& P_X(Z_n \leq t) - \Phi(t) \\
& = -\phi(t) \left\{ \frac{H_1(t)}{3!\sqrt{n}} \left(\frac{E_X(X_1 - \mu)^3}{\sigma_X^3} \right) \right. \\
& + \frac{H_2(t)}{4!n} \left(\frac{E_X(X_1 - \mu)^4}{\sigma_X^4} - 3 \right) + \frac{H_3(t)}{2(3!)^2n} \left(\frac{E_X(X_1 - \mu)^3}{\sigma_X^3} \right)^2 \Big\} + o(1/n).
\end{aligned} \tag{3.13}$$

In view of (3.12), and also (3.10), when the data and the weights have four moments and the data satisfy Cramér's condition, we conclude that for both Methods I and II, when $|\text{SRF}^w(\theta)|$ is small, the CLT for $g_n^w(\theta)$ becomes more accurate. In particular, when $|\text{SRF}^w(\theta)|$ is negligible then the CLT for $g_n^w(\theta)$ is second order accurate, i.e., of order $O(1/n)$. In contrast, by virtue of (3.13), and also (3.11), one can readily see that, under the same conditions for the data, the CLT for t_n is only first order accurate, i.e., of order $O(1/\sqrt{n})$.

4 Confidence intervals for μ based on $g_n^w(\theta)$

In view of Methods I and II, we are now to put the refinement provided by the randomized pivots $g_n^w(\theta)$, as in (2.2), to use by constructing more accurate confidence intervals for the population mean μ , in the case of univariate data. In this section we also study the length of these confidence intervals.

The use of $g_n^w(\theta)$ as a pivot results in asymptotic $100(1 - \alpha)\%$, $0 < \alpha < 1$, size confidence intervals for μ of the form:

$$\mathcal{C}^w(\theta) = [\min\{M_n, N_n\}, \max\{M_n, N_n\}], \quad (4.1)$$

where

$$\begin{aligned} M_n &= \frac{-z_{1-\alpha/2} S_n \sqrt{n E_w(w_1 - \theta)^2} - \sum_{i=1}^n (w_i - \theta) X_i}{-\sum_{i=1}^n (w_i - \theta)}, \\ N_n &= \frac{z_{1-\alpha/2} S_n \sqrt{n E_w(w_1 - \theta)^2} - \sum_{i=1}^n (w_i - \theta) X_i}{-\sum_{i=1}^n (w_i - \theta)}, \end{aligned}$$

and $z_{1-\alpha/2}$ is the $100(1 - \alpha/2)$ th percentile of the standard normal distribution.

We now examine the length of $\mathcal{C}^w(\theta)$ which is

$$\text{Length}(\mathcal{C}^w(\theta)) := \frac{2z_{1-\alpha/2} S_n}{|\sum_{i=1}^n (w_i - \theta)| / \sqrt{n E_w(w_1 - \theta)^2}}. \quad (4.2)$$

It is easy to see that, for $\mathcal{C}^w(\theta)$ when it is constructed by the means of Method I, since $\theta \neq E_w(w_1)$, as $n \rightarrow +\infty$, we have

$$\text{Length}(\mathcal{C}^w(\theta)) = o_{P_{X,w}}(1). \quad (4.3)$$

In other words, choosing the weights and their associated window constants in accordance with Method I, to create the randomized pivot $g_n^w(\theta)$, as in (2.2), results in confidence intervals for μ whose lengths approach zero, as the sample size increases.

On the other hand, in the scenario of Method II we have $\theta = E_w(w_1)$. The latter choice of θ implies that, as $n \rightarrow +\infty$, for all $b \in \mathbb{R}$

$$P_w\left(\sum_{i=1}^n (w_i - \theta) / \sqrt{n E_w(w_1 - \theta)^2} \leq b\right) \rightarrow \Phi(b).$$

The preceding CLT for the weights, in view of (4.2), implies that, as $n \rightarrow +\infty$

$$P_{X,w}(\text{Length}(\mathcal{C}^w(\theta)) \leq \ell) \rightarrow P(2\sigma_X z_{1-\alpha/2} / |Z| \leq \ell), \quad (4.4)$$

where $\ell \in \mathbb{R}$ and Z is a standard normal random variable.

Remark 4.1. *In view of (4.4), the length of a confidence interval based on the pivot $g_n^w(\theta)$, when it is constructed in accordance with Method II, converges in distribution to a scaled inverse of a folded standard normal random variable rather than shrinking, while, as it was seen in Section 3, this method results in CLTs for $g_n^w(\theta)$ that, under appropriate conditions, are second order accurate (cf. (3.10), (3.12) and Table 3), recalling that in Method II, $SRF^w(\theta) = 0$.*

4.1 Numerical examples for Methods I and II

In this section we present some numerical results to illustrate the refinement provided by the randomized confidence intervals $\mathcal{C}^w(\theta)$, as in (4.1), when the random weights and their associated window constants are chosen in accordance with Methods I and II. In addition to examining the accuracy in terms of empirical probabilities of coverage, here, we also address the length of the randomized confidence intervals $\mathcal{C}^w(\theta)$.

In our numerical studies in Tables 1-3 below, we generate 1000 randomized confidence intervals as in (4.1), with nominal size of 95%, using the cut-off points $\pm z_{1-\alpha/2} = \pm 1.96$ therein, and 1000 classical t -confidence intervals $\mathcal{E} := \bar{X}_n \pm 1.96 S_n / \sqrt{n}$, based on the same generated data with the same nominal size and cut-off points.

In Tables 1-3 $\text{coverage}(\mathcal{C}^w(\theta))$ and $\text{length}(\mathcal{C}^w(\theta))$ stand, respectively, for the empirical probabilities of coverage and the empirical lengths of the generated confidence intervals $\mathcal{C}^w(\theta)$. Also, $\text{coverage}(\mathcal{E})$ and $\text{length}(\mathcal{E})$ stand, respectively, for the empirical probabilities of coverage and the empirical lengths of the generated t -confidence intervals \mathcal{E} with nominal size 95%.

In the following Tables 1-2, under the scenario of Method I, we examine the higher accuracy provided by the randomized pivot $g_n^w(\theta)$, as in (2.2), over the classical t_n , as in (2.1).

Table 1: $w \stackrel{d}{=} \chi^2(7)$, $\theta = 9.3$, $\text{SRF}^w(9.3) \approx -0.662$ and nominal size 95%

	n	coverage($\mathcal{C}^w(9.3)$)	length($\mathcal{C}^w(9.3)$)	coverage(\mathcal{E})	length(\mathcal{E})
$X \stackrel{d}{=} \text{Binomial}(10, 0.1)$	10	0.933	5.545	0.905	1.153
	20	0.947	2.308	0.921	0.810
	30	0.950	1.522	0.935	0.672
$X \stackrel{d}{=} \text{Poisson}(1)$	10	0.931	5.447	0.908	1.204
	20	0.943	2.096	0.928	0.861
	30	0.945	1.518	0.933	0.705
$X \stackrel{d}{=} \text{Lognormal}(0, 1)$	10	0.897	8.027	0.801	2.147
	20	0.907	4.829	0.855	1.608
	30	0.930	2.973	0.875	1.343
$X \stackrel{d}{=} \text{Exponential}(1)$	10	0.913	6.753	0.873	1.144
	20	0.933	2.740	0.903	0.839
	30	0.940	1.617	0.920	0.694
$X \stackrel{d}{=} \chi^2(1)$	10	0.890	7.772	0.833	1.552
	20	0.917	3.363	0.878	1.159
	30	0.927	2.158	0.895	0.957
$X \stackrel{d}{=} \text{Beta}(5, 1)$	10	0.926	0.834	0.894	0.167
	20	0.940	0.336	0.923	0.121
	30	0.946	0.227	0.929	0.099

Table 2: $w \stackrel{d}{=} \text{Bernoulli}(1/3)$, $\theta = 0.58$, $\text{SRF}^w(0.58) \approx -0.7$ and nominal size 95%

	n	coverage($\mathcal{C}^w(0.58)$)	length($\mathcal{C}^w(0.58)$)	coverage(\mathcal{E})	length(\mathcal{E})
$X \stackrel{d}{=} \text{Binomial}(10, 0.1)$	10	0.941	4.446	0.913	1.140
	20	0.946	2.452	0.928	0.832
	30	0.950	1.841	0.934	0.673
$X \stackrel{d}{=} \text{Poisson}(1)$	10	0.942	4.597	0.905	1.235
	20	0.947	2.618	0.927	0.861
	30	0.949	1.906	0.929	0.708
$X \stackrel{d}{=} \text{Lognormal}(0, 1)$	10	0.897	8.227	0.808	2.118
	20	0.921	4.859	0.849	1.604
	30	0.932	3.730	0.870	1.369
$X \stackrel{d}{=} \text{Exponential}(1)$	10	0.928	4.415	0.868	1.142
	20	0.938	2.552	0.904	0.840
	30	0.945	1.882	0.914	0.696
$X \stackrel{d}{=} \chi^2(1)$	10	0.909	5.993	0.836	1.562
	20	0.926	3.464	0.876	1.150
	30	0.937	2.630	0.900	0.966
$X \stackrel{d}{=} \text{Beta}(5, 1)$	10	0.937	0.661	0.895	0.167
	20	0.942	0.368	0.923	0.121
	30	0.948	0.264	0.927	0.099

Remark 4.2. From Tables 1 and 2, it is evident that the randomized pivots $g_n^w(\theta)$, as in (2.2), when constructed according to Method I, can significantly outperform

t_n , as in (2.1), in terms of accuracy.

In the following Table 3 we examine numerically the performance of $g_n^w(\theta)$ when it is constructed based on Method II.

Table 3: $w \stackrel{d}{=} \text{Normal}(0, 1)$, $\theta = 0$, $\text{SRF}^w(0) = 0$ and nominal size 95%

	n	coverage($\mathcal{C}^w(0)$)	length($\mathcal{C}^w(0)$)	coverage(\mathcal{E})	length(\mathcal{E})
$X \stackrel{d}{=} \text{Binomial}(10, 0.1)$	10	0.963	18.744	0.892	1.170
	20	0.954	14.497	0.922	0.824
	100	0.949	17.197	0.948	0.372
$X \stackrel{d}{=} \text{Poisson}(1)$	10	0.951	28.539	0.899	1.245
	20	0.948	22.211	0.933	0.874
	100	0.954	29.994	0.946	0.391
$X \stackrel{d}{=} \text{Lognormal}(0, 1)$	10	0.957	24.953	0.894	1.217
	20	0.951	49.103	0.84	1.724
	100	0.947	41.609	0.909	0.822
$X \stackrel{d}{=} \text{Exponential}(1)$	10	0.944	30.549	0.87	1.235
	20	0.956	27.110	0.902	0.870
	100	0.953	19.068	0.943	0.389
$X \stackrel{d}{=} \chi^2(1)$	10	0.937	31.819	0.844	1.712
	20	0.946	34.559	0.865	1.242
	100	0.947	32.376	0.924	0.554
$X \stackrel{d}{=} \text{Beta}(5, 1)$	10	0.948	3.098	0.891	0.175
	20	0.950	2.984	0.932	0.122
	100	0.952	3.254	0.935	0.054

Note that in Table 3, as the sample size increases, the lengths of the confidence intervals $\mathcal{C}^w(0)$, as in (4.1) with $\theta = 0$ therein, that are constructed based on Method II, fluctuate rather than shrink (see (4.4)).

5 Randomized pivots with higher accuracy using triangular random weights

In this section we put the scenario of Method I into perspective, and extend it to also include triangular weights. The idea here is to relate the size of the given sample to the random weights.

In this section, we let $w^{(n)}, w_1^{(n)}, \dots, w_n^{(n)}$ be a *triangular* array of random weights that is *independent* from the data X, X_1, \dots, X_n . The random weights $w^{(n)}$ here, can *either* be an i.i.d. array of random variables with $E_w |w_1^{(n)}|^3 < +\infty$, *or* they can have a \mathcal{M} ultinomial distribution with size \mathcal{K}_n , i.e.,

$$(w_1^{(n)}, \dots, w_n^{(n)}) \stackrel{d}{=} \mathcal{M}ultinomial(\mathcal{K}_n; p_{1,n}, \dots, p_{n,n}), \quad (5.1)$$

where $\mathcal{K}_n = \sum_{i=1}^n w_i^{(n)} \rightarrow +\infty$, as $n \rightarrow +\infty$ and $\sum_{i=1}^n p_{i,n} = 1$.

We are now to introduce Method I.1, as a generalization of Method I, that can yield asymptotically, in n , SRF's whose absolute values are small or negligible.

Method I.1: Let $w_1^{(n)}, \dots, w_n^{(n)}$ be as above. Choose a real valued constant θ^* in such a way that for given δ , so that $|\delta|$ can be arbitrary small or zero,

- (i) $\theta^* \neq \lim_{n \rightarrow +\infty} E_w(w_1^{(n)})$ and
- (ii) $\lim_{n \rightarrow +\infty} \text{SRF}^{w^{(n)}}(\theta^*) := \lim_{n \rightarrow +\infty} \frac{E_w(w_1^{(n)} - \theta^*)^3}{(E_w(w_1^{(n)} - \theta^*)^2)^{3/2}} = \delta,$

Moreover, as $n \rightarrow +\infty$, θ^* should also satisfy the following maximal negligibility condition.

$$(iii) \max_{1 \leq i \leq n} (w_i^{(n)} - \theta^*)^2 / n = o_{P_w}(1). \quad (5.2)$$

The counterpart of the pivot $g_n^w(\theta)$, as in (2.2), in the context of Method I.1 is the following $g_n^{w^{(n)}}(\theta^*)$ which is defined as:

$$g_n^{w^{(n)}}(\theta^*) := \sum_{i=1}^n (w_i^{(n)} - \theta^*)(X_i - \mu) / \left(S_n \sqrt{n E_w(w_1^{(n)} - \theta^*)^2} \right). \quad (5.3)$$

We note that one can, equivalently, replace $n E_w(w_1^{(n)} - \theta^*)^2$, in the denominator of $g_n^{w^{(n)}}(\theta^*)$, by $\sum_{j=1}^n (w_j^{(n)} - \theta^*)^2$.

Remark 5.1. *The maximal negligibility condition (5.2) is a sufficient condition for the following CLT, for all $t \in \mathbb{R}$.*

$$P_{X,w}(g_n^{w^{(n)}}(\theta^*) \leq t) \rightarrow \Phi(t), \text{ as } n \rightarrow +\infty. \quad (5.4)$$

The preceding CLT is valid even when the random sample X_1, \dots, X_n has only two moments provided that (5.2) holds true.

The CLT in (5.4) is a consequence of the well known Lindeberg-Feller CLT in a conditional sense. We further elaborate on the CLT in (5.4) by noting that, in light of the dominated convergence theorem, (5.4) follows from the following conditional CLT:

As $n \rightarrow +\infty$, for all $t \in \mathbb{R}$, (5.2) suffices to have

$$P_{X|w}(g_n^{w^{(n)}}(\theta^*) \leq t) \rightarrow \Phi(t) \text{ in probability} - P_w,$$

where $P_{X|w}$ stands for the conditional probability of X given the weights $w_1^{(n)}, \dots, w_n^{(n)}$.

It is noteworthy that a typical condition under which (5.2) holds true is when the identically distributed triangular weights $w_1^{(n)}$'s, for each n , have a finite k th moment, where $k \geq 3$, and $\lim_{n \rightarrow +\infty} E_w |w_1^{(n)} - \theta^*|^k = c$, for some positive constant c . The validity of the latter claim can be investigated by an application of Markov's inequality for $nP_w(|w_1^{(n)} - \theta|/\sqrt{n} > \epsilon)$, where ϵ is an arbitrary positive number.

5.1 On the \mathcal{M} ultinomial random weights

We now consider a particular form of the \mathcal{M} ultinomial distribution (5.1), in which $\mathcal{K}_n = n$ and $p_{i,n} = 1/n$ for $1 \leq i \leq n$, i.e.,

$$(w_1^{(n)}, \dots, w_n^{(n)}) \stackrel{d}{=} \mathcal{M}ultinomial(n; 1/n, \dots, 1/n). \quad (5.5)$$

On taking $\theta^* = 1.32215$, for example, in Method I.1, when the weights are \mathcal{M} ultinomially distributed as in (5.5), the randomized pivot $g_n^{w^{(n)}}(\theta^*)$, as in (5.3), assumes the following specific form:

$$g_n^{w^{(n)}}(1.32215) = \sum_{i=1}^n (w_i^{(n)} - 1.32215)(X_i - \mu) / \left(S_n \sqrt{n E_w (w_1^{(n)} - 1.32215)^2} \right). \quad (5.6)$$

The window constant $\theta^* = 1.32215$, in view of Method I.1, when the weights are \mathcal{M} ultinomial as in (5.5), was obtained from the following three steps:

Step 1: Obtain the general form of $\text{SRF}^{w^{(n)}}(\theta)$ in this case as follows:

$$\begin{aligned} \text{SRF}^{w^{(n)}}(\theta) &= \frac{E_w (w_1^{(n)} - \theta)^3}{(E_w (w_1^{(n)} - \theta)^2)^{3/2}} \\ &= \frac{-\theta^3 + 3\theta^2 - 3\theta(n(n-1)/n^2 + 1) + n(n-1)(n-2)/n^3 + 3n(n-1)/n^2 + 1}{(\theta^2 - 2\theta + n(n-1)/n^2 + 1)^{3/2}}. \end{aligned} \quad (5.7)$$

Step 2: Obtain the limit of $\text{SRF}^{w^{(n)}}(\theta)$, that was derived in Step 1, as follows:

$$\lim_{n \rightarrow +\infty} \text{SRF}^{w^{(n)}}(\theta) = (-\theta^3 + 3\theta^2 - 6\theta + 5) / (\theta^2 - 2\theta + 2)^{3/2}$$

Step 3: In light of Step 2, for $\theta^* = 1.32215$, $\lim_{n \rightarrow +\infty} \text{SRF}^{w^{(n)}}(1.32215)$ assumes a value approximately equal to $\delta = 0.0001$ which is negligible.

We note that the maximal negligibility condition (5.2) holds for the \mathcal{M} ultinomial weights as in (5.5). The latter is true since, in this case, we have $\lim_{n \rightarrow +\infty} E_w (w_1^{(n)} - 1.32215)^4 = c$, where c is a positive number whose value is not specified here (cf. the

paragraph following Remark 5.1). By this, we conclude that, on taking $\theta^* = 1.32215$, all the assumptions in Method I.1 hold true for the \mathcal{M} ultinomial weights, as in (5.5).

In the present context of \mathcal{M} ultinomial weights, the $100(1 - \alpha)\%$ confidence intervals for μ , based on the pivot $g_n^{w^{(n)}}(1.32215)$, as in (5.6), follow the general form (4.1). However, the fact that here we have the constrain $\sum_{i=1}^n w_i^{(n)} = n$, enables us to specify (4.1) for μ in this context, as follows:

$$C^{w^{(n)}}(1.32215) := \frac{\sum_{i=1}^n (w_i^{(n)} - 1.32215)X_i \pm z_{1-\alpha/2}S_n \sqrt{nE_w(w_1^{(n)} - 1.32215)^2}}{0.32215n}. \quad (5.8)$$

\mathcal{M} ultinomial random variables, of the form (5.5), also appear in the area of the weighted bootstrap, also known as the generalized bootstrap (cf., for example, Arenal-Gutiérrez and Matrán [1], Barbe and Bertail [2], Csörgő *et al.* [8], Mason and Newton [15] and references therein), where they represent the count of the number of times each observation is selected in a re-sampling with replacement from a given sample. Motivated by this, somewhat remote, relation between the bootstrap and our randomized approach in Method I.1, when the weights are as in (5.5), we are now to conduct a numerical comparison between the two methods. After some further elaborations on the weighted bootstrap, we present our numerical results in Table 4 below.

To explain the viewpoint of the weighted bootstrap, we first consider a bootstrap sample X_1^*, \dots, X_n^* that is drawn with replacement from the original sample X_1, \dots, X_n . Observe now that for the bootstrap sample mean $\bar{X}_n^* := \sum_{k=1}^n X_k^*/n$ we have

$$\bar{X}_n^* = \sum_{i=1}^n w_i^{(n)} X_i / n,$$

where, for each i , $1 \leq i \leq n$, $w_i^{(n)}$ is the count of the number of times the index i of X_i was selected. It is easy to observe that the weights $(w_1^{(n)}, \dots, w_n^{(n)})$ are \mathcal{M} ultinomially distributed, as in (5.5), and they are independent from the data X_1, \dots, X_n .

To conduct our numerical comparisons, we consider the bootstrap t -confidence intervals (cf. Efron and Tibshirani [11]) that are generally known to be efficient of the second order in probability- P_X (cf., for example, Hall [12], Shao and Tu [19] and Singh [22]). To construct a bootstrap t -confidence interval for the population mean μ , first a large number, say B , of independent bootstrap samples of size n are drawn from the original data. Let us represent them by $X_1^*(b), \dots, X_n^*(b)$, where $1 \leq b \leq B$. The bootstrap version of t_n , as in (2.1), is computed for each one of these B bootstrap sub-samples to have $t_n^*(1), \dots, t_n^*(B)$, where

$$\begin{aligned}
t_n^* &:= \sqrt{n}(\bar{X}_n^* - \bar{X}_n)/S_n^* \\
&= \sum_{i=1}^n (w_i^{(n)} - 1)X_i/(\sqrt{n}S_n^*),
\end{aligned}$$

S_n^{*2} is the bootstrap sample variance and $w^{(n)}$'s are as in (5.5). These B bootstrap t -statistics are then sorted in ascending order to have $t_n^*[1] \leq \dots \leq t_n^*[B]$. When, for example, $B = 1000$, a bootstrap t -confidence interval for μ with the nominal size 95% is constructed by setting:

$$\mathcal{C}^* := t_n^*[25] \leq t_n \leq t_n^*[975].$$

For the same nominal size of 95%, we are now to compare the performance of the randomized confidence interval $\mathcal{C}^{w^{(n)}}(1.32215)$, as in (5.8), to that of the bootstrap t -confidence interval \mathcal{C}^* , in Table 4 below.

In Table 4, we generate 1000 confidence intervals $\mathcal{C}^{w^{(n)}}(1.32215)$. To do so, we use 1000 replications of the data sets X_1, \dots, X_n , and the \mathcal{M} ultinomial weights $(w_1^{(n)}, \dots, w_n^{(n)})$, as in (5.5). For each one of the generated data sets, based on $B = 1000$ bootstrap samples, we also generate 1000 bootstrap t -confidence intervals \mathcal{C}^* , with nominal size of 95%.

Similarly to our setups for Tables 1-3, in Table 4, we let $\text{coverage}(\mathcal{C}^{w^{(n)}}(1.32215))$ and $\text{length}(\mathcal{C}^{w^{(n)}}(1.32215))$ stand for the empirical coverage probabilities and the empirical lengths of the therein generated randomized confidence intervals $\mathcal{C}^{w^{(n)}}(1.32215)$. Also, in Table 4, we let $\text{coverage}(\mathcal{C}^*)$ and $\text{length}(\mathcal{C}^*)$ stand for the empirical probabilities of coverage and the empirical lengths of the bootstrap confidence intervals \mathcal{C}^* .

The relatively close performance, in terms of accuracy, of the bootstrap t -confidence intervals with $B = 1000$ bootstrap samples, and the randomized pivot $g_n^{w^{(n)}}(1.32215)$, as in (5.6), in Table 4 is interesting. Further refinements to the randomization approach Method I.1 that results in randomized pivots that can outperform, in terms of accuracy, Method I.1 are presented in Method I.2 in Subsection 5.2 below.

It is worth noting that the class of \mathcal{M} ultinomial random weights (5.1) is far richer than the particular form (5.5). Our focus on the latter was mainly the result of its application in the area of the weighted bootstrap. Clearly different choices of the size \mathcal{K}_n and/or $p_{i,n}$ in (5.1) yield different randomizing weights.

Remark 5.2. *The use of the randomized pivots introduced in this paper to construct confidence intervals for the mean by no means is computationally intensive, while the bootstrap is a computationally demanding method. Also, using the randomization methods discussed in this paper, one does not have to deal with the problem of how large the number of bootstrap replications B , should be. Moreover, the error reduction*

Table 4: $w^{(n)}$ are as in (5.5), $\theta^* = 1.32215$, $\text{SRF}^{w^{(n)}}(1.32215) \approx 10^{-4}$ and nominal size 95%

	n	coverage($C^{w^{(n)}}(1.32215)$)	length($C^{w^{(n)}}(1.32215)$)	coverage(C^*)	length(C^*)
$X \stackrel{d}{=} \text{Binomial}(10, 0.1)$	13	0.943	3.137	0.971	Inf
	20	0.948	2.570	0.943	0.887
	30	0.950	2.147	0.951	0.702
$X \stackrel{d}{=} \text{Poisson}(1)$	13	0.958	3.290	0.972	Inf
	20	0.952	2.712	0.967	0.929
	30	0.946	2.248	0.946	0.731
$X \stackrel{d}{=} \text{Lognormal}(0, 1)$	10	0.923	6.940	0.903	5.609
	20	0.937	5.115	0.921	2.748
	30	0.946	4.296	0.931	1.828
$X \stackrel{d}{=} \text{Exponential}(1)$	10	0.950	3.451	0.937	1.637
	20	0.952	2.626	0.941	1.073
	30	0.953	2.225	0.952	0.769
$X \stackrel{d}{=} \chi^2(1)$	10	0.925	4.688	0.938	3.222
	20	0.944	3.684	0.946	1.661
	30	0.950	3.060	0.946	1.236
$X \stackrel{d}{=} \text{Beta}(5, 1)$	10	0.943	0.511	0.946	0.238
	20	0.951	0.381	0.943	0.134
	30	0.953	0.313	0.956	0.108

methods introduced in this paper enable one to easily trace down the effect of the randomization on the length of the confidence intervals in the univariate case, and the volume of the randomized confidence rectangles when the data are multivariate (cf. (4.1), Section 6 and Appendix I).

It is also worth noting that the randomization framework allows regulating the error of an inference by choosing a desired value for the SRF. This can be done by choosing the random weights from a virtually unlimited class, as characterized in the above Method I, Method II, Method I.1 and also Method I.2 below.

5.2 Fixed sample approach to higher accuracy using triangular random weights

The approach discussed in Method I.1 considers triangular random weights, to tie the random weights to the sample size, and chooses the window constant θ^* therein in such a way that it makes the absolute value of the SRF arbitrarily small, in the limit. Here, we also consider the triangular random weights as described at the beginning of this section and introduce a method to increase the accuracy of the CLT based inferences about the mean for fixed sample sizes.

For each fixed sample size n , the following Method I.2 yields a further sharpening of the asymptotic refinement provided by Method I.1 and it reads as follows:

Method I.2: Let the weights $w^{(n)}$'s be as described right above Method I.1. If

for a given δ , so that $|\delta|$ can be arbitrary small or zero, there exist a real value θ^* so that for the weights $w^{(n)}$'s, we have

$$(i) \theta^* \neq \lim_{n \rightarrow +\infty} E_w(w_1^{(n)}),$$

$$(ii) \lim_{n \rightarrow +\infty} \text{SRF}^{w^{(n)}}(\theta^*) = \lim_{n \rightarrow +\infty} \frac{E_w(w_1^{(n)} - \theta^*)^3}{(E_w(w_1^{(n)} - \theta^*)^2)^{3/2}} = \delta \text{ and}$$

$$(iii) \max_{1 \leq i \leq n} (w_i^{(n)} - \theta^*)^2 / n = o_{P_w}(1),$$

then, for each n , choose a real valued constant θ_n in such a way that it satisfies the following conditions (iv) and (v).

$$(iv) \theta_n \neq E_w(w_1^{(n)}),$$

$$(v) \text{SRF}^{w^{(n)}}(\theta_n) := \frac{E_w(w_1^{(n)} - \theta_n)^3}{(E_w(w_1^{(n)} - \theta_n)^2)^{3/2}} = \delta.$$

The viewpoint in Method I.2, in principle, requires choosing different θ_n for different sample sizes n , for a given δ . Also, it is not difficult to see that Method I.1 is the asymptotic version of Method I.2.

Under the scenario of Method I.2, after choosing an appropriate window value θ_n , for a given δ , we define the randomized pivot $g_n^{w^{(n)}}(\theta_n)$ as follows:

$$g_n^{w^{(n)}}(\theta_n) := \sum_{i=1}^n (w_i^{(n)} - \theta_n)(X_i - \mu) / \left(S_n \sqrt{n E_w(w_1^{(n)} - \theta_n)^2} \right). \quad (5.9)$$

The normalizing sequence $n E_w(w_1^{(n)} - \theta_n)^2$ in the denominator of $g_n^{w^{(n)}}(\theta_n)$ can, equivalently, be replaced by $\sum_{j=1}^n (w_j^{(n)} - \theta_n)^2$.

We note that, for each fixed n and given δ , when $|\delta|$ is small, Method I.2 and its associated pivots $g_n^{w^{(n)}}(\theta_n)$, as in (5.9), yield higher accuracy than those that result from the use of Method I.1 and its associated pivots $g_n^{w^{(n)}}(\theta^*)$, as in (5.3). This is true since, in Method I.2, the window constants θ_n are tailored for each fixed n to make $\text{SRF}^{w^{(n)}}(\theta_n) = \delta$. This is in contrast to the viewpoint of Method I.1 in which the therein defined skewness reducing factor $\text{SRF}^{w^{(n)}}(\theta^*)$ assumes the given value δ in the limit.

Despite their differences in the context of finite samples, both Method I.1 and Method I.2 yield randomized pivots, as in (5.3) and (5.9), that can outperform their classical counterpart t_n , as in (2.1), in terms of accuracy (see Tables 4 above and also Tables 5 and 6 below).

Under the scenario of Method I.2, the confidence intervals for μ based on the randomized pivots $g_n^{w^{(n)}}(\theta_n)$, also admit the general form (4.1), only with $w^{(n)}$ in place of w and θ_n in place of θ therein. Hence, in the following numerical studies we denote them by $\mathcal{C}^{w^{(n)}}(\theta_n)$.

In order to illustrate the refinement provided by Method I.2, we consider random samples of sizes $n = 10$ and $n = 20$ from the heavily skewed Lognormal(0,1). We also consider \mathcal{M} ultinomially distributed weights as in (5.5). Choosing the random weights here to be \mathcal{M} ultinomially distributed, as in (5.5) is so that the numerical results in Tables 5 and 6 below should be comparable to their counterparts in Table 4 above where the data have a Lognormal(0,1) distribution.

On taking $\delta = 10^{-4}$ in Method I.2, we saw in Subsection 5.1, that for $\theta^* = 1.32215$ we have

$$\lim_{n \rightarrow +\infty} \text{SRF}^{w^{(n)}}(1.32215) = \lim_{n \rightarrow +\infty} \frac{E_w(w_1^{(n)} - 1.32215)^3}{(E_w(w_1^{(n)} - 1.32215)^2)^{3/2}} \approx 10^{-4} \text{ and}$$

$$\max_{1 \leq i \leq n} (w_i^{(n)} - 1.32215)^2 / n = o_{P_w}(1).$$

Recall that for the \mathcal{M} ultinomial weights, as in (5.5), the general form of $\text{SRF}^{w^{(n)}}(\theta)$ was already derived in (5.7). In view of the latter result, it is easy to check that when $n = 10$, on taking $\theta_{10} = 1.2601$ we have $\text{SRF}^{w^{(10)}}(1.2601) \approx 10^{-4}$. Also, for $n = 20$, taking $\theta_{20} = 1.29129$ yields $\text{SRF}^{w^{(20)}}(1.29129) \approx 10^{-4}$.

Consider now $\mathcal{C}^{w^{(10)}}(1.2601)$ and $\mathcal{C}^{w^{(20)}}(1.29129)$, the confidence intervals for μ of nominal size 95% based on Method I.2 and samples of size $n = 10$ and $n = 20$, which result, respectively, from setting:

$$-1.96 \leq g_{10}^{w^{(10)}}(1.2601) = \frac{\sum_{i=1}^{10} (w_i^{(10)} - 1.2601)(X_i - \mu)}{S_{10} \sqrt{10 E_w(w_1^{(10)} - 1.2601)^2}} \leq 1.96,$$

$$-1.96 \leq g_{20}^{w^{(20)}}(1.29129) = \frac{\sum_{i=1}^{20} (w_i^{(20)} - 1.29129)(X_i - \mu)}{S_{20} \sqrt{20 E_w(w_1^{(20)} - 1.29129)^2}} \leq 1.96.$$

In the following Tables 5 and 6 we generate 1000 replications of Lognormal(0,1) data and \mathcal{M} ultinomial weights, as in (5.5), for $n = 10$ and $n = 20$. We let $\text{coverage}(\mathcal{C}^{w^{(10)}}(1.2601))$ and $\text{coverage}(\mathcal{C}^{w^{(20)}}(1.29129))$ stand for the respective empirical probabilities of coverage of $\mathcal{C}^{w^{(10)}}(1.2601)$ and $\mathcal{C}^{w^{(20)}}(1.29129)$. We also let $\text{length}(\mathcal{C}^{w^{(10)}}(1.2601))$ and $\text{length}(\mathcal{C}^{w^{(20)}}(1.29129))$ stand for the respective empirical lengths of $\mathcal{C}^{w^{(10)}}(1.2601)$ and $\mathcal{C}^{w^{(20)}}(1.29129)$.

Table 5: $n = 10$, $\theta^{(10)} = 1.2601$, $\text{SRF}^{w^{(10)}}(1.2601) \approx 10^{-4}$ and nominal size 95%

	coverage($\mathcal{C}^{w^{(10)}}(1.2601)$)	length($\mathcal{C}^{w^{(10)}}(1.2601)$)
X $\stackrel{d}{=}$ Lognormal(0, 1)	0.936	8.23

In comparison between the two methods Method I.2 and Method I.1, the former can outperform the latter, for the same weights and the same δ (see Tables 5 and

Table 6: $n = 20$, $\theta^{(20)} = 1.29129$, $\text{SRF}^{w^{(20)}}(1.29129) \approx 10^{-4}$ and nominal size 95%

	$\text{coverage}(\mathcal{C}^{w^{(20)}}(1.29129))$	$\text{length}(\mathcal{C}^{w^{(20)}}(1.29129))$
$X \stackrel{d}{=} \text{Lognormal}(0, 1)$	0.944	5.646

6, and compare them to their counterparts in Table 4 in which Method I.1 and the bootstrap are examined).

6 Randomized multivariate pivots

The skewness reducing methods introduced in Methods I, II, I.1 and I.2, can be extended to the case when the data are multidimensional.

In this section, we first restrict our attention to Method I and extend it to address multivariate data (see also Remark 6.3 below, where Methods I.1 or I.2 are used to randomize multivariate data). We show how the randomization technique of Method I can result in more accurate multivariate CLT's.

To state our results in this section, we let $\mathbb{X}_j = (X_{1,j}, \dots, X_{p,j})'$, $1 \leq j \leq n$, be independent copies of a p -variate, $p \geq 1$, random vector $\mathbb{X} = (X_1, \dots, X_p)'$ such that, for some $k \geq 3$, $E_{\mathbb{X}}\|\mathbb{X}_1\|^k < +\infty$, where $\|\mathbb{X}\| = (\sum_{s=1}^p X_s^2)^{1/2}$. Furthermore, we let $\mu = E(\mathbb{X}) = (\mu_1, \dots, \mu_p)'$ and Σ be the theoretical mean and the theoretical covariance matrix of the data \mathbb{X} . Moreover, for throughout use in this section, we assume that the covariance matrix Σ is positive definite.

We now define the pivotal quantity $\mathbb{G}_n^{(w)}(\theta)$ that is the multidimensional version of $g_n^w(\theta)$, as in (2.2), as follows:

$$\mathbb{G}_n^{(w)}(\theta) := \left(\frac{\mathbf{S}_n^{-1/2}}{\sqrt{nE_w(w_1 - \theta)^2}} \right) \sum_{i=1}^n (w_i - \theta)(\mathbb{X}_i - \mu), \quad (6.1)$$

where the *univariate* random weights w , that are independent from the data $\mathbb{X}_1, \dots, \mathbb{X}_n$, and the window constant θ are as characterized in Method I in Section 2, and $\mathbf{S}_n^{-1/2}$ is the inverse of a positive definite square root of the $(p \times p)$ sample covariance matrix

$$\mathbf{S}_n = \sum_{j=1}^n (\mathbb{X}_j - \bar{\mathbb{X}}_n)(\mathbb{X}_j - \bar{\mathbb{X}}_n)' / (n - 1), \quad (6.2)$$

where $\bar{\mathbb{X}}_n = \sum_{j=1}^n \mathbb{X}_j$.

The multivariate pivotal quantity $\mathbb{G}_n^{(w)}(\theta)$ is a randomized version of the classical multivariate t -statistic

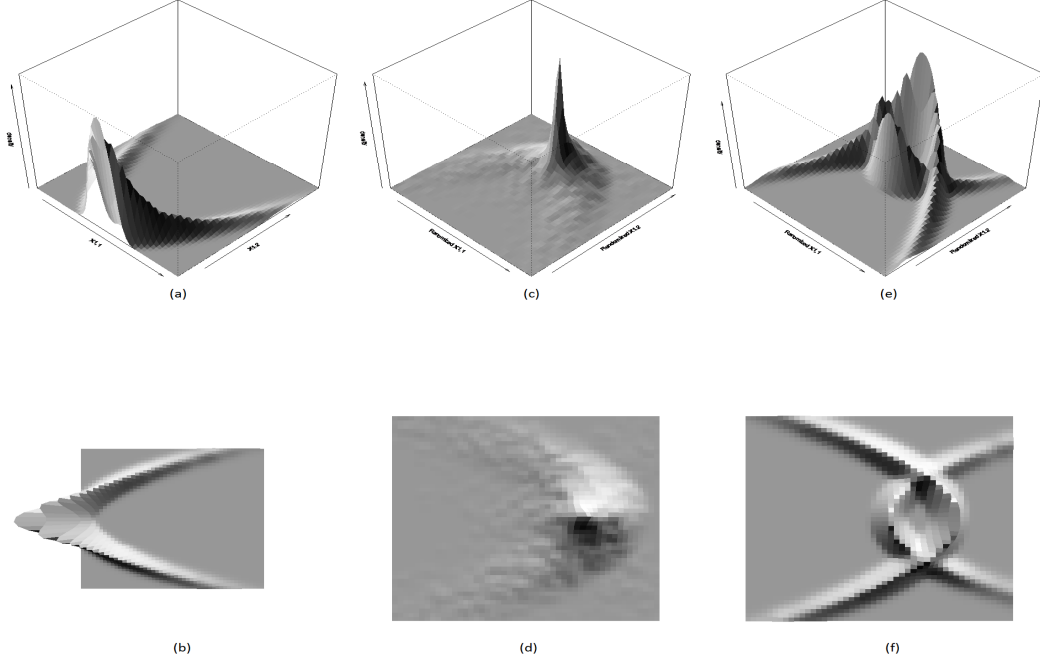


Figure 2: (Illustration of the effect of Method I on bivariate data)

Panels (a) and (b): Two views of the density plot of the original i.i.d. data $(X_{i,1}, X_{i,2})'$, $1 \leq i \leq 20000$, where $X_{i,1} \stackrel{d}{=} \text{Normal}(0, 1)$, $X_{i,2} = X_{i,1}^2$, with empirical Mardia's Measure of skewness, cf. Appendix II, equal to 13.209. Panels (c) and (d): Two views of the density plot of the randomized data $(w_i - 9.3)(X_{i,1}, X_{i,2})'$, $1 \leq i \leq 20000$, where $w_i \stackrel{d}{=} \chi^2(7)$, with empirical Mardia's Measure of skewness equal to 3.239. Panels (e) and (f): Two views of the density plot of the randomized data $(w_i - 0.58)(X_{i,1}, X_{i,2})'$, $1 \leq i \leq 20000$, where $w_i \stackrel{d}{=} \text{Bernoulli}(1/3)$, with empirical Mardia's Measure of skewness equal to 6.216.

$$\mathbb{T}_n := \mathbf{S}_n^{-1/2} \sum_{i=1}^n (\mathbb{X}_i - \mu) / \sqrt{n}. \quad (6.3)$$

Remark 6.1. *The possibility of lack of invertibility of \mathbf{S}_n is a minor drawback that can be resolved by replacing \mathbf{S}_n by an asymptotically equivalent extended versions of it that are invertible, for all n . This idea is due to Sepanski [18], who proposed replacing \mathbf{S}_n by \mathbf{D}_n that can have either one of the following two forms:*

$$\mathbf{D}_n = \mathbf{S}_n + l_n \mathbf{I} \quad (6.4)$$

$$\mathbf{D}_n = \begin{cases} \mathbf{S}_n, & \text{if } \mathbf{S}_n \text{ is invertible;} \\ \mathbf{I}, & \text{otherwise,} \end{cases} \quad (6.5)$$

where \mathbf{I} is the identity matrix on \mathbb{R}^p and l_n , in (6.4), is a sequence of positive numbers that can approach zero arbitrary fast. Hence, when \mathbf{S}_n is not invertible, it can be replaced by either one of the two forms of \mathbf{D}_n , as in (6.4) and (6.5), in both $\mathbb{G}_n^{(w)}(\theta)$ and \mathbb{T}_n as in (6.1) and (6.3), respectively.

In order to show that Method I continues to yield smaller error for the CLT for the randomized multidimensional pivot $\mathbb{G}_n^{(w)}(\theta)$, we first consider weights and data with a finite fourth moments, i.e., when $E_w|w_1|^4 < +\infty$ and $E_{\mathbb{X}}\|\mathbb{X}_1\|^4 < +\infty$. The refinement provided by (6.1) under the milder condition that the data and the weights have a finite third moment will be discussed later on in this section.

We replace the sample covariance matrix \mathbf{S}_n by the limiting covariance matrix Σ . To do so, we adopt the component-wise convergence in probability, and almost surely, definition of a sequence of random matrices. More precisely, we say a sequence of random matrices \mathbf{A}_n , $n \geq 1$, converges in probability, or almost surely, to the random matrix \mathbf{A} if each component of \mathbf{A}_n converges in probability, or almost surely, to its counterpart in \mathbf{A} . This definition, in turn, enables one to conclude that replacing the sample covariance matrix \mathbf{S}_n by the limiting covariance matrix Σ , when $E_{\mathbb{X}}\|\mathbb{X}_1\|^4 < +\infty$, results in an error of magnitude $o(1/n)$. The latter statement is true since both the sample variances and the sample covariances approach their theoretical counterparts at the rate of $o(1/n)$ (see (3.9)). Consequently, the multivariate pivot \mathbb{T}_n agrees in distribution with

$$\mathbb{Z}_n := \frac{\Sigma^{-1/2}}{\sqrt{n}} \sum_{i=1}^n (\mathbb{X}_i - \mu) \quad (6.6)$$

up to an error of order $o(1/n)$ where, $\Sigma^{-1/2}$ is the square root of the inverse of the limiting covariance matrix Σ .

Consider now the standardized data

$$\mathbb{Y}_i = (Y_{i,1}, \dots, Y_{i,p})' := \Sigma^{-1/2}(\mathbb{X}_i - \mu), \quad 1 \leq i \leq n, \quad (6.7)$$

and denote the distribution function of \mathbb{Z}_n by $F_{n,\mathbb{X}}(t_1, \dots, t_p)$, where $(t_1, \dots, t_p) \in \mathbb{R}^p$. Moreover, let $\Phi(t_1, \dots, t_p)$ and $\phi(t_1, \dots, t_p)$ be the respective distribution and density functions of a p -variate standard normal evaluated at (t_1, \dots, t_p) . Also, for the ease of notation, we define

$$\int_{\prod_{s=1}^p (-\infty, t_s]} [\dots] := \int_{-\infty}^{t_p} \dots \int_{-\infty}^{t_1} [\dots] dt_1 \dots dt_p.$$

Under the assumptions of Theorem 19.2 of Bhattacharya and Rao [7], for all $(t_1, \dots, t_p) \in \mathbb{R}^p$, we have

$$\begin{aligned}
F_{n,\mathbb{X}}(t_1, \dots, t_p) &= \Phi(t_1, \dots, t_p) \\
&+ \sum_{j=1}^p \frac{E(Y_{1,j}^3)}{\sqrt{n}} \int_{\prod_{s=1}^p (-\infty, t_s]} -1/6(-t_j^3 + 3t_j)\phi(t_1, \dots, t_p) \\
&+ \sum_{1 \leq j \neq k \leq p} \frac{E(Y_{1,j}^2 Y_{1,k})}{\sqrt{n}} \int_{\prod_{s=1}^p (-\infty, t_s]} -1/2(t_j^2 t_k + t_k)\phi(t_1, \dots, t_p) \\
&+ \sum_{\substack{1 \leq j, k, l \leq p \\ j \neq k, k \neq l, l \neq j}} \frac{E(Y_{1,j} Y_{1,k} Y_{1,l})}{\sqrt{n}} \int_{\prod_{s=1}^p (-\infty, t_s]} -t_j t_k t_l \phi(t_1, \dots, t_p) \\
&+ O(1/n).
\end{aligned} \tag{6.8}$$

As for the randomized i.i.d. data $(w_i - \theta)(\mathbb{X}_i - \mu) = ((w_i - \theta)(X_{1,i} - \mu_1), \dots, (w_i - \theta)(X_{p,i} - \mu_p))'$, let $F_{n,w,\mathbb{X}}(t_1, \dots, t_p)$ stand for their distribution function for all $(t_1, \dots, t_p) \in \mathbb{R}^p$. Consider now the randomized multivariate quantity

$$\begin{aligned}
\mathbb{Z}_n^w(\theta) &:= \frac{\Sigma_{w,\mathbb{X}}^{-1/2}}{\sqrt{n}} \sum_{i=1}^n (w_i - \theta)(\mathbb{X}_i - \mu) \\
&= \frac{\Sigma^{-1/2}}{\sqrt{n E_w(w_1 - \theta)^2}} \sum_{i=1}^n (w_i - \theta)(\mathbb{X}_i - \mu),
\end{aligned} \tag{6.9}$$

where $\Sigma_{w,\mathbb{X}}$ and Σ , respectively, are the covariance matrices of the randomized data $(w_i - \theta)(\mathbb{X}_i - \mu)$ and the original ones \mathbb{X}_i .

An argument similar to the one used to show the asymptotic equivalence of \mathbb{T}_n and \mathbb{Z}_n , as in (6.3) and (6.6), enables us to also conclude that $\mathbb{Z}_n^w(\theta)$, as in (6.9), is asymptotically equivalent to the randomized pivot $\mathbb{G}_n^w(\theta)$, as defined in (6.1), at the rate of $o(1/n)$.

By virtue of the above setup we now can write the counterpart of the Edgeworth expansion (6.8), under the same conditions on the data \mathbb{X}_i , for the randomized quantity $\mathbb{Z}_n^w(\theta)$ as follows:

$$\begin{aligned}
F_{n,w,\mathbb{X}}(t_1, \dots, t_p) &= \Phi(t_1, \dots, t_p) \\
&+ \sum_{j=1}^p \frac{E_{w,\mathbb{X}}\left(\frac{(w_1-\theta)^3}{E_w^{3/2}(w_1-\theta)^2} Y_{1,j}^3\right)}{\sqrt{n}} \int_{\prod_{s=1}^p (-\infty, t_s]} -1/6(-t_j^3 + 3t_j)\phi(t_1, \dots, t_p) \\
&+ \sum_{1 \leq j \neq k \leq p} \frac{E_{w,\mathbb{X}}\left(\frac{(w_1-\theta)^3}{E_w^{3/2}(w_1-\theta)^2} Y_{1,j}^2 Y_{1,k}\right)}{\sqrt{n}} \int_{\prod_{s=1}^p (-\infty, t_s]} -1/2(t_j^2 t_k + t_k)\phi(t_1, \dots, t_p) \\
&+ \sum_{\substack{1 \leq j,k,l \leq p \\ j \neq k, k \neq l, l \neq j}} \frac{E_{w,\mathbb{X}}\left(\frac{(w_1-\theta)^3}{E_w^{3/2}(w_1-\theta)^2} Y_{1,j} Y_{1,k} Y_{1,l}\right)}{\sqrt{n}} \int_{\prod_{s=1}^p (-\infty, t_s]} -t_j t_k \phi(t_1, \dots, t_p) \\
&+ O(1/n).
\end{aligned}$$

Due to independence of the data \mathbb{X}_i , and their standardized versions \mathbb{Y}_i , as in (6.7), from the random weights w_i , the preceding Edgewroth expansion is equivalent to the following relation.

$$\begin{aligned}
F_{n,w,\mathbb{X}}(t_1, \dots, t_p) &= \Phi(t_1, \dots, t_p) \\
&+ \text{SRF}^w(\theta) \left\{ \sum_{j=1}^p \frac{E(Y_{1,j}^3)}{\sqrt{n}} \int_{\prod_{s=1}^p (-\infty, t_s]} -1/6(-t_j^3 + 3t_j)\phi(t_1, \dots, t_p) \right. \\
&+ \sum_{1 \leq j \neq k \leq p} \frac{E(Y_{1,j}^2 Y_{1,k})}{\sqrt{n}} \int_{\prod_{s=1}^p (-\infty, t_s]} -1/2(t_j^2 t_k + t_k)\phi(t_1, \dots, t_p) \\
&+ \left. \sum_{\substack{1 \leq j,k,l \leq p \\ j \neq k, k \neq l, l \neq j}} \frac{E(Y_{1,j} Y_{1,k} Y_{1,l})}{\sqrt{n}} \int_{\prod_{s=1}^p (-\infty, t_s]} -t_j t_k \phi(t_1, \dots, t_p) \right\} \\
&+ O(1/n).
\end{aligned} \tag{6.10}$$

Denoting now the distribution functions of the multidimensional pivots \mathbb{T}_n and $\mathbb{G}_n^w(\theta)$, respectively, by $Q_n(t_1, \dots, t_p)$ and $Q_{n,w,\mathbb{X}}(t_1, \dots, t_p)$, from (6.8), as $n \rightarrow +\infty$, we conclude that

$$Q_n(t_1, \dots, t_p) - \Phi(t_1, \dots, t_p) = O(1/\sqrt{n}), \tag{6.11}$$

while, under the same conditions on the data \mathbb{X} , the expansion (6.10), as $n \rightarrow +\infty$, yields

$$Q_{n,w,\mathbb{X}}(t_1, \dots, t_p) - \Phi(t_1, \dots, t_p) = (\text{SRF}^w(\theta))O(1/\sqrt{n}) + O(1/n). \tag{6.12}$$

By virtue of Method I, on choosing appropriate random weights w and a window constant θ to construct $\mathbb{G}_n^{(w)}(\theta)$, as in (6.1), one can achieve CLTs with error rates up to $O(1/n)$. The optimal rate of $O(1/n)$ is achieved when θ is chosen in such a way that $|\text{SRF}^w(\theta)|$ is negligible. This result is in contrast to the error rate of $O(1/\sqrt{n})$, as in (6.11), that is the error rate of the CLT for \mathbb{T}_n , as in (6.3), that cannot be improved upon without restricting the class of the distributions of the original data to the symmetrical ones.

Under the milder assumption that the data and the weights have a finite third moment, in view of Theorem 19.2 of Bhattacharya and Rao [7], using a similar argument as the one used to derive (6.12), one can conclude the following statement which is the counterparts of (6.12) in this context.

$$Q_{n,w,\mathbb{X}}(t_1, \dots, t_p) - \Phi(t_1, \dots, t_p) = (\text{SRF}^w(\theta))O(1/\sqrt{n}) + o(1/\sqrt{n}). \quad (6.13)$$

Once again, a comparison between (6.13) and (6.11) shows that, on assuming that $E\|\mathbb{X}\|^3 < +\infty$ and $E_w|w_1|^3 < +\infty$, the CLT for the randomized pivot $\mathbb{G}_n^{(w)}(\theta)$, as in (6.1), when constructed under the scenario of Method I, will have smaller error as compared to that of \mathbb{T}_n , as in (6.3). In particular, when $|\text{SRF}^w(\theta)|$ is set to be negligible, the CLT for $\mathbb{G}_n^{(w)}(\theta)$ is accurate of order $o(1/\sqrt{n})$ rather than $O(1/\sqrt{n})$, as in (6.11), that is the error rate of the CLT for \mathbb{T}_n , as in (6.3).

Remark 6.2. *The effect of the skewness reduction technique of Method I on the volume of the simultaneous p -dimensional confidence rectangles for the vector valued mean $\mu = (\mu_1, \dots, \mu_p)'$, can be addressed by its effect on the marginal confidence intervals for each of the mean components μ_i , $1 \leq i \leq p$. The latter effect is essentially the same as that discussed in details in Section 4, in case of univariate data. For details on the effect of randomization on the volume of the randomized confidence rectangles, we refer to Appendix I.*

The results in Tables 7-12 below are based on 1000 replications of the therein specified bivariate data and the random weights. As for the cut-off points, we used ± 2.2365 in Tables 7-12, since $P((Z_1, Z_2) \in [-2.2365, 2.2365]^2) \approx 0.95$, where (Z_1, Z_2) has a standard bivariate normal distribution.

Tables 7-10 are numerical comparisons between the performance of the randomized pivot $\mathbb{G}_n^{(w)}(\theta)$, as in (6.1), when constructed according to Method I, and that of the classical \mathbb{T}_n , as in (6.3).

Table 7: $w \stackrel{d}{=} \chi^2(7)$, $\theta = 9.3$, $\text{SRF}^w(9.3) \approx -0.662$ and nominal size 95%

$\mathbb{X} = (X, X^2)'$	n	coverage of $\mathbb{G}_n^{(w)}(9.3)$	coverage of \mathbb{T}_n
$X \stackrel{d}{=} \text{Normal}(0, 1)$	30	0.920	0.863
	50	0.933	0.884
	100	0.945	0.921
$X \stackrel{d}{=} \text{Exponential}(1)$	100	0.909	0.841
	300	0.930	0.895
	400	0.940	0.903

Table 8: $w \stackrel{d}{=} \text{Bernoulli}(1/3)$, $\theta = 0.58$, $\text{SRF}^w(0.58) \approx -0.7$ and nominal size 95%

$\mathbb{X} = (X, X^2)'$	n	coverage of $\mathbb{G}_n^{(w)}(0.58)$	coverage of \mathbb{T}_n
$X \stackrel{d}{=} \text{Normal}(0, 1)$	30	0.913	0.845
	50	0.936	0.894
	100	0.948	0.917
$X \stackrel{d}{=} \text{Exponential}(1)$	100	0.925	0.835
	300	0.939	0.897
	400	0.949	0.912

In Tables 9 and 10 below, the i.i.d. vector valued data $\mathbb{X} = (\eta_1, \eta_2)'$ consist of the first two terms of the moving average process $\eta_t = \zeta_t + 0.2\zeta_{t-1}$, $t \geq 1$, where $E(\zeta_s) = 0$, for $s \geq 0$.

Table 9: $w \stackrel{d}{=} \chi^2(7)$, $\theta = 9.3$, $\text{SRF}^w(9.3) \approx -0.662$ and nominal size 95%

$\mathbb{X} = (\eta_1, \eta_2)'$	n	coverage of $\mathbb{G}_n^{(w)}(9.3)$	coverage of \mathbb{T}_n
$\zeta \stackrel{d}{=} \text{Normal}(0, 1)$	10	0.927	0.875
	20	0.948	0.915
$\zeta \stackrel{d}{=} \text{Exponential}(1)-1$	30	0.932	0.885
	50	0.943	0.917

Table 10: $w \stackrel{d}{=} \text{Bernoulli}(1/3)$, $\theta = 0.58$, $\text{SRF}^w(0.58) \approx -0.7$ and nominal size 95%

$\mathbb{X} = (\eta_1, \eta_2)'$	n	coverage of $\mathbb{G}_n^{(w)}(0.58)$	coverage of \mathbb{T}_n
$\zeta \stackrel{d}{=} \text{Normal}(0, 1)$	10	0.935	0.870
	20	0.950	0.912
$\zeta \stackrel{d}{=} \text{Exponential}(1)-1$	30	0.945	0.892
	50	0.950	0.917

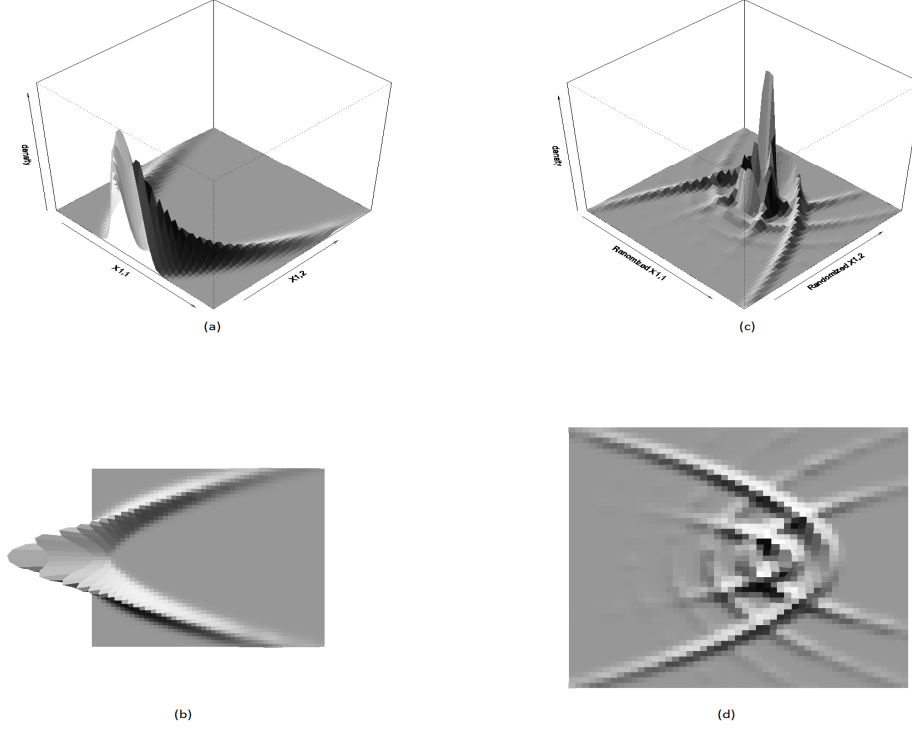


Figure 3: (Illustration of the effect of Method I.1 on bivariate data)

Panels (a) and (b): Two views of the density plot of the data $(X_{i,1}, X_{i,2})'$, $1 \leq i \leq 20000$, where $X_{i,1} \stackrel{d}{=} \text{Normal}(0,1)$ and $X_{i,2} = X_{i,1}^2$. Panels(c) and (d): Two views of the density plot of the randomized data $(w_i^{(n)} - 1.32215)(X_{i,1}, X_{i,2})'$, $1 \leq i \leq 20000$, where $(w_1^{(n)}, \dots, w_{20000}^{(n)}) \stackrel{d}{=} \mathfrak{M}\text{ultinomial}(20000; 1/20000, \dots, 1/20000)$.

Remark 6.3. *In addition to Method I that was discussed in this section, in the case of multivariate data, Methods I.1 and I.2, as stated in Section 5 can also result in significant refinements when they are used to construct the multidimensional randomized pivot $\mathbb{G}_n^{(w)}(\theta)$, as in (6.1), with $w^{(n)}$ in place of w therein.*

We demonstrate the validity of Remark 6.3 numerically in Tables 11 and 12. To establish the results in Table 12, we use Method I.1 with the weights having $\mathcal{M}\text{ultinomial}$ distribution as in (5.5), and $\mathbb{X} = (\eta_1, \eta_2)'$ in Table 12, are as in Tables 9 and 10.

Table 11: $w^{(n)}$ are as in (5.5), $\theta^* = 1.32215$, $\text{SRF}^{w^{(n)}}(1.32215) \approx 10^{-4}$ and nominal size 95%

$\mathbb{X} = (X, X^2)'$	n	coverage of $\mathbb{G}_n^{(w)}(1.32215)$	coverage of \mathbb{T}_n
$X \stackrel{d}{=} \text{Normal}(0, 1)$	30	0.933	0.857
	50	0.941	0.889
	100	0.950	0.917
$X \stackrel{d}{=} \text{Exponential}(1)$	100	0.926	0.860
	300	0.940	0.894
	400	0.947	0.906

Table 12: $w^{(n)}$ are as in (5.5), $\theta^* = 1.32215$, $\text{SRF}^{w^{(n)}}(1.32215) \approx 10^{-4}$ and nominal size 95%

$\mathbb{X} = (\eta_1, \eta_2)'$	n	coverage of $\mathbb{G}_n^{(w)}(1.32215)$	coverage of \mathbb{T}_n
$\zeta \stackrel{d}{=} \text{Normal}(0, 1)$	10	0.955	0.870
	20	0.951	0.901
$\zeta \stackrel{d}{=} \text{Exponential}(1)-1$	30	0.944	0.893
	50	0.951	0.922

Appendix I: Asymptotically exact size randomized confidence rectangles

In the case of multivariate data, the effect of the randomization methods discussed in Section 6, on the volume of the resulting randomized (hyper) confidence rectangles can be studied by looking at the marginal confidence intervals for each component of the mean vector. To further elaborate on the idea, for simplicity we restrict our attention to two dimensional data as the idea is the same for data with higher dimensions. Furthermore, here, we borrow the notation used in Section 6, and note that we first consider the randomization approach of Method I. The effect of the other randomization methods on the volume of the resulting randomized confidence rectangles are to be addressed later on.

Consider the i.i.d. bivariate data $\mathbb{X}_j = (X_{1,j}, X_{2,j})'$, $1 \leq j \leq n$, with mean $\mu = (\mu_1, \mu_2)'$. Furthermore, for ease of notation, let $\mathbf{S}_n^{-1/2} =: \begin{bmatrix} a_n & b_n \\ b_n & c_n \end{bmatrix}$, where \mathbf{S}_n , as defined in (6.2) with $p = 2$, is the sample covariance matrix.

The classical $100(1 - \alpha)\%$ confidence rectangle for $\mu = (\mu_1, \mu_2)'$ based on the pivot \mathbb{T}_n , as in (6.3), is as follows:

$$\left[\sum_{j=1}^n X_{1,j}/n \pm z_\alpha^* \left(\frac{b_n + c_n}{a_n c_n - b_n^2} \right) / \sqrt{n} \right] \times \left[\sum_{j=1}^n X_{2,j}/n \pm z_\alpha^* \left(\frac{b_n + a_n}{a_n c_n - b_n^2} \right) / \sqrt{n} \right], \quad (\star)$$

where $P(-z_\alpha^* \leq Z_1 \leq z_\alpha^* \cap -z_\alpha^* \leq Z_2 \leq z_\alpha^*) = 1 - \alpha$, and (Z_1, Z_2) has a standard bivariate normal distribution, i.e., $(Z_1, Z_2) \stackrel{d}{=} \text{Normal}((0, 0)', \mathbf{I})$.

The area of the confidence rectangle (\star) is

$$L_{n,\mathbb{X}} := (2z_\alpha^*)^2 \frac{(b_n + c_n)(b_n + a_n)}{n(a_n c_n - b_n^2)^2}.$$

Observe now that, as $n \rightarrow +\infty$, under the moment conditions assumed for the data in Section 6, we have $L_{n,\mathbb{X}} = o_{P_{\mathbb{X}}}(1)$.

The randomized version of the confidence rectangle (\star) for $\mu = (\mu_1, \mu_2)'$, in view of Method I, and based on the randomized pivot $\mathbb{G}_n^{(w)}(\theta)$, as defined in (6.1), is of the following form:

$$\left[\min\{M_{1,n}, N_{1,n}\}, \max\{M_{1,n}, N_{1,n}\} \right] \times \left[\min\{M_{2,n}, N_{2,n}\}, \max\{M_{2,n}, N_{2,n}\} \right], \quad (\star\star)$$

where

$$\begin{aligned} M_{1,n} &= \frac{\sum_{j=1}^n (w_j - \theta) X_{1,j}}{\sum_{i=1}^n (w_i - \theta)} - z_\alpha^* \left(\frac{b_n + c_n}{a_n c_n - b_n^2} \right) \left(\frac{\sqrt{n E_w(w_1 - \theta)^2}}{\sum_{i=1}^n (w_i - \theta)} \right), \\ N_{1,n} &= \frac{\sum_{j=1}^n (w_j - \theta) X_{1,j}}{\sum_{i=1}^n (w_i - \theta)} + z_\alpha^* \left(\frac{b_n + c_n}{a_n c_n - b_n^2} \right) \left(\frac{\sqrt{n E_w(w_1 - \theta)^2}}{\sum_{i=1}^n (w_i - \theta)} \right), \\ M_{2,n} &= \frac{\sum_{j=1}^n (w_j - \theta) X_{2,j}}{\sum_{i=1}^n (w_i - \theta)} - z_\alpha^* \left(\frac{b_n + a_n}{a_n c_n - b_n^2} \right) \left(\frac{\sqrt{n E_w(w_1 - \theta)^2}}{\sum_{i=1}^n (w_i - \theta)} \right), \\ N_{2,n} &= \frac{\sum_{j=1}^n (w_j - \theta) X_{2,j}}{\sum_{i=1}^n (w_i - \theta)} + z_\alpha^* \left(\frac{b_n + a_n}{a_n c_n - b_n^2} \right) \left(\frac{\sqrt{n E_w(w_1 - \theta)^2}}{\sum_{i=1}^n (w_i - \theta)} \right). \end{aligned}$$

The area of the randomized confidence rectangle $(\star\star)$ has the following form:

$$L_{n,\mathbb{X},w}(\theta) := (2z_\alpha^*)^2 \left(\frac{(b_n + c_n)(b_n + a_n)}{(a_n c_n - b_n^2)^2} \right) \left(\frac{1}{\sum_{i=1}^n (w_i - \theta) / \sqrt{n E_w(w_1 - \theta)^2}} \right)^2.$$

Hence, similarly to the univariate case, in case of multidimensional data, under the conditions of Section 6, in view of Method I, as $n \rightarrow +\infty$, we have $L_{n,\mathbb{X},w}(\theta) = o_{P_{\mathbb{X},w}}(1)$. In other words, Method I yields randomized confidence regions for the mean vector, that shrink as the sample size increases.

We remark that, in the multivariate case, Methods I.1 and I.2 also yield randomized confidence rectangles of the form $(\star\star)$, with the notation w therein replaced by $w^{(n)}$, that shrink as the sample size increases. A similar argument to the one used to derive (4.4) shows that the latter conclusion concerning the shrinkage of the randomized confidence regions, in view of Methods I, I.1 and I.2, does not hold true when the randomized pivot $\mathbb{G}_n^{(w)}(\theta)$ is constructed using Method II.

Appendix II: The effect of Method I on Mardia's measure of skewness

A number of definitions for the concept of skewness of multivariate data can be found in the literature when the assumption of normality is dropped. Mardia's characteristics of skewness for multivariate data, cf. Mardia [14], is, perhaps, the most popular in the literature. This measures of skewness is valid when the covariance matrix of the distribution is nonsingular. For further discussions and developments on Mardia's skewness and kurtosis characteristics, we refer to Kollo [13] and references therein.

Mardia's measure of skewness for p -variate distributions is defined as follows:

$$\beta_{\mathbb{X},p} := E_{\mathbb{X}}\{(\mathbb{X}_1 - \mu)' \Sigma^{-1} (\mathbb{X}_2 - \mu)\}^3,$$

where \mathbb{X}_1 and \mathbb{X}_2 are i.i.d. and Σ^{-1} is the inverse of the invertible covariance matrix Σ .

The following reasoning shows how small values of $|\text{SRF}^w(\theta)|$, as in Method I, result in smaller values for Mardia's measure of skewness for the randomized vectors $(w - \theta)(\mathbb{X} - \mu)$ as compared to that of \mathbb{X} .

Let $\beta_{\mathbb{X},w,p}$ be Mardia's measure of skewness of the randomized data $(w - \theta)(\mathbb{X} - \mu)$ and write

$$\begin{aligned} \beta_{\mathbb{X},w,p} &= E_{w,\mathbb{X}}\left\{\frac{(w_1 - \theta)(w_2 - \theta)}{E_w(w_1 - \theta)^2} (\mathbb{X}_1 - \mu)' \Sigma^{-1} (\mathbb{X}_2 - \mu)\right\}^3 \\ &= \frac{E_w^2(w_1 - \theta)^3}{E_w^3(w_1 - \theta)^2} E_{\mathbb{X}}\{(\mathbb{X}_1 - \mu)' \Sigma^{-1} (\mathbb{X}_2 - \mu)\}^3 \\ &= \left(\text{SRF}^w(\theta)\right)^2 \beta_{\mathbb{X},p}, \end{aligned}$$

where $(w_1 - \theta)\mathbb{X}_1$ and $(w_2 - \theta)\mathbb{X}_2$ are i.i.d. with respect to the joint distribution $P_{\mathbb{X},w}$. The preceding relation shows that employing Method I enables one to make Mardia's characteristic of skewness arbitrarily small.

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